CS 58500 – Theoretical Computer Science Toolkit

Lecture 18 (04/14)

Boolean Function Analysis (III)

https://ruizhezhang.com/course_spring_2026.html

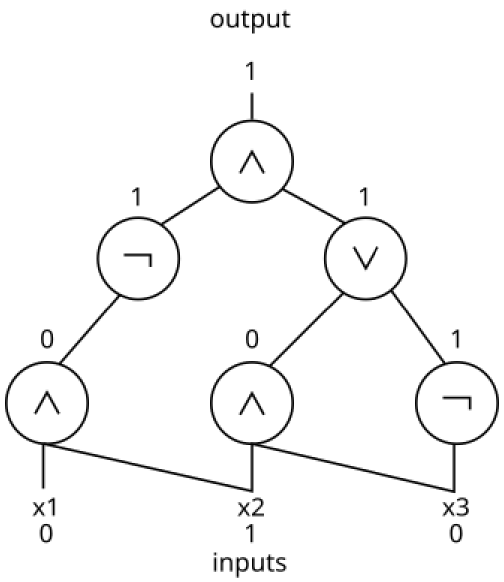
Today's Lecture

- Complexity of Boolean Functions
- Query Complexity vs. xyz
- Randomized Query Complexity
- The Sensitivity Theorem

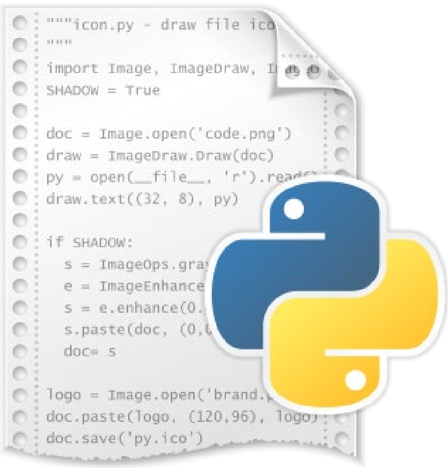
Complexity of Boolean Functions

Let $f\colon\{0,1\}^n \to \mathbb{R}$ be a Boolean function. How do we measure the complexity of f?

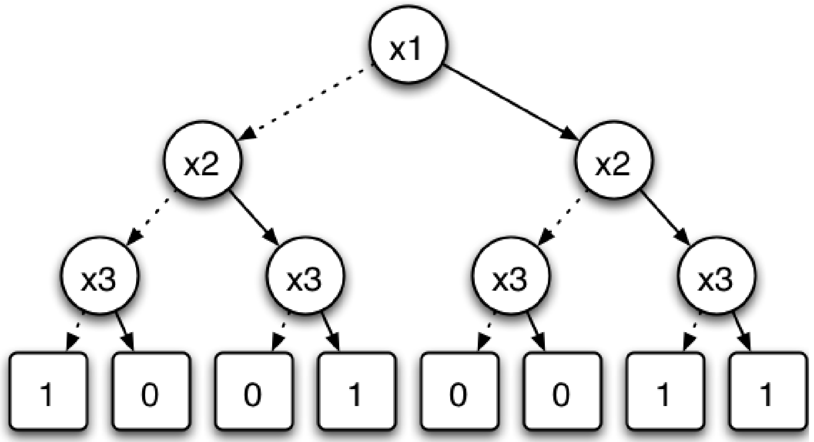
- For simplicity, we assume $f\colon\{0,1\}^n \to \{0,1\}$, i.e., a decision problem with n-bit input and "YES/NO" output. (You are encouraged to convert the results in this lecture to functions like $f\colon\{0,1\}^n \to \{-1,1\}$)



Circuit complexity



Kolmogorov complexity



Query complexity

Complexity of Boolean Functions

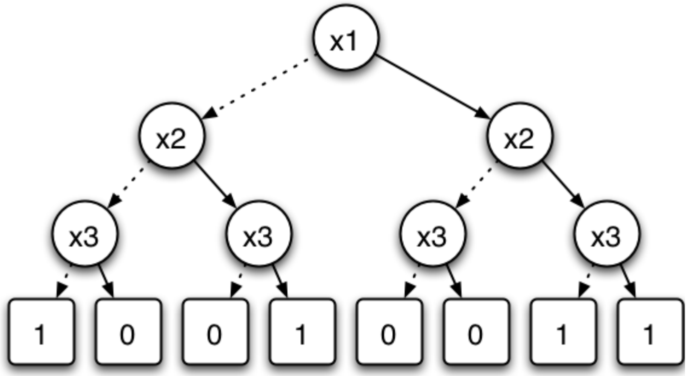
Theorem (Shannon '42**).** With probability at least $1 - o(1)$, a random function $f: \{0,1\}^n \to \{0,1\}$ requires a circuit (with bounded fan-in) of size $\Omega(2^n/n)$

Theorem (Lupanov '52**).** Any function $f: \{0,1\}^n \to \{0,1\}$ can be computed by a circuit using $\left(1 + \mathrm{o}(1)\right) 2^n/n$ gates of fan-in ≤ 2

Complexity of Boolean Functions

Query complexity is also called the decision tree complexity

- A decision tree over variables $x_1, \ldots, x_n$ is a binary tree where each internal node has two children, left and right
- Each internal node is labelled with a variable, and each leaf is labelled with a value of 0 or 1
- To evaluate a decision tree at a point $x \in \{0,1\}^n$, we start from the root, and at each internal node with label x_i we query the value of x_i, go left if $x_i = 0$ and right if $x_i = 1$ until we reach a leaf. The leaf's value is the decision tree's output on x
- For a Boolean function $f: \{0,1\}^n \to \{0,1\}$, we let $\mathrm{D}(f)$ denote the smallest depth of a decision tree computing f
- Trivially, $0 \leq \mathrm{D}(f) \leq n$



Complexity of Boolean Functions

The goal of this lecture is to use analytical measures of a Boolean function to characterize its query complexity

- Degree $\deg(f)$
- Certificate complexity $\mathrm{C}(f)$
- Sensitivity $\mathrm{s}(f)$
- Block sensitivity $\mathrm{bs}(f)$

Today's Lecture

- Complexity of Boolean Functions
- Query Complexity vs. xyz
- Randomized Query Complexity
- The Sensitivity Theorem

Query Complexity vs. Degree

Recall the Fourier transform of $f: \{0,1\}^n \to \{0,1\}$:

$$f(x) = \sum_S \hat{f}(S)\chi_S(x)$$

We can also use the monomial basis $\{\prod_{i\in S} x_i\}$ to expand f:

$$f(x) = \sum_S a_S \prod_{i\in S} x_i$$

How to use express a_S in terms of f?

- By inclusion-exclusion formula, $a_S = \sum_{T\subseteq S}(-1)^{|S\setminus T|} f(\mathbf{1}_T) \quad \Longrightarrow \; a_S \in \mathbb{Z}$

How to use express a_S in terms of $\hat{f}$?

$$a_S = (-2)^{|S|} \sum_{T\supseteq S} \hat{f}(T)$$

Query Complexity vs. Degree

For $f:\{0,1\}^n \to \{0,1\}$ with

$$f(x) = \sum_S \hat{f}(S)\chi_S(x) = \sum_S a_S \prod_{i \in S} x_i$$

we define the (real) degree of f as the

$$\deg(f) \coloneqq \max\{|S| : a_S \neq 0\} = \max\{|S| : \hat{f}(S) \neq 0\}$$

Example:

$$\text{Parity}(x) = x_1 + \cdots + x_n \bmod 2 = \frac{1 - \chi_{\mathbf{1}}(x)}{2}$$

- $\deg(\text{Parity}) = n$
- Its degree as a polynomial in $\mathbb{F}_2$ is 1

Query Complexity vs. Degree

Proposition. For every $f\colon\{0,1\}^n \to \{0,1\}$, we have $\deg(f) \leq \mathrm{D}(f)$

Proof.

- If f can be expressed as a degree-d multilinear polynomial $\{0,1\}^n \to \mathbb{R}$, then $\deg(f) \leq d$
- Consider the decision tree that computes f with depth $\mathrm{D}(f)$
- Let $\mathcal{L}_1$ be the set of leaves with label 1. And for each $x \in \{0,1\}^n$, let $\ell(x)$ be the leaf reached by the tree when computing $f(x)$

$$f(x) = \sum_{\ell \in \mathcal{L}_1} \mathbf{1}_{\ell(x)=\ell}$$

- For every leaf $\ell \in \mathcal{L}_1$, there is a unique path from the root to ℓ. Let T_ℓ and F_ℓ denote the indices of variables on this path that returned the value 1 and 0, respectively

Query Complexity vs. Degree

Proposition. For every $f: \{0,1\}^n \to \{0,1\}$, we have $\deg(f) \le \mathrm{D}(f)$

Proof.

$$f(x) = \sum_{\ell \in \mathcal{L}_1} \mathbf{1}_{\ell(x)=\ell}$$

- For every leaf $\ell \in \mathcal{L}_1$, there is a unique path from the root to ℓ. Let T_ℓ and F_ℓ denote the indices of variables on this path that returned the value 1 and 0, respectively

$$\mathbf{1}_{\ell(x)=\ell} \quad \Leftrightarrow \quad \left(\prod_{i \in T_\ell} x_i\right)\left(\prod_{j \in F_\ell} (1 - x_j)\right)$$

$$f(x) = \sum_{\ell \in \mathcal{L}_1} \left(\prod_{i \in T_\ell} x_i\right)\left(\prod_{j \in F_\ell} (1 - x_j)\right)$$

- $\deg(f) \le \max_{\ell \in \mathcal{L}_1} (|T_\ell| + |F_\ell|) = \mathrm{D}(f)$

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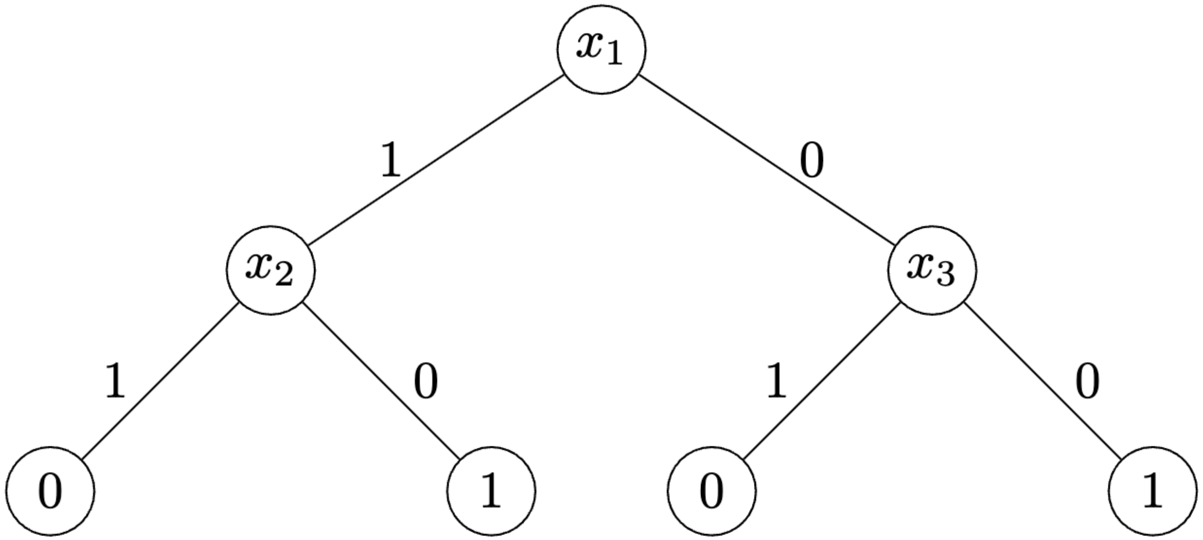
Query Complexity vs. Degree



$$f(x_1, x_2, x_3) = x_1(1 - x_2) + (1 - x_1)(1 - x_3)$$
$$= (x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge \neg x_3)$$

Query Complexity vs. Certificate Complexity

A certificate for an input $x \in \{0,1\}^n$ to a Boolean function $f: \{0,1\}^n \to \{0,1\}$ is a set $S \subseteq [n]$ of indices such that f is constant on all inputs that match x_S

- $\mathrm{C}_f(x)$ denotes the size of a smallest certificate for x
- The certificate complexity $\mathrm{C}(f) \coloneqq \max_x \mathrm{C}_f(x)$

Example:

- $\mathrm{C}(\mathrm{Parity}) = n$
- $\mathrm{C}(\mathrm{AND}) = n$
- If f is a k-junta, then $\mathrm{C}(f) = k$

Query Complexity vs. Certificate Complexity

Theorem. For every $f:\{0,1\}^n \to \{0,1\}$, we have $\mathrm{C}(f) \le \mathrm{D}(f) \le \mathrm{C}(f)^2$

Proof.

- If there is a decision tree of depth $D(f)$ that computes f, then for any input x, the set of variables queried by the decision tree determines $f(x)$, which forms a certificate for x. Thus, $\mathrm{C}(f) \le \mathrm{D}(f)$
- To prove $\mathrm{D}(f) \le \mathrm{C}(f)^2$, observe that if $S \subseteq [n]$ is a certificate for $x \in f^{-1}(0)$ and $T \subseteq [n]$ is a certificate for $y \in f^{-1}(1)$, then $S \cap T \ne \emptyset$
- Define

$$\mathrm{C}^0(f) \coloneqq \max_{x \in f^{-1}(0)} \mathrm{C}_f(x) \le \mathrm{C}(f), \qquad \mathrm{C}^1(f) \coloneqq \max_{x \in f^{-1}(1)} \mathrm{C}_f(x) \le \mathrm{C}(f)$$

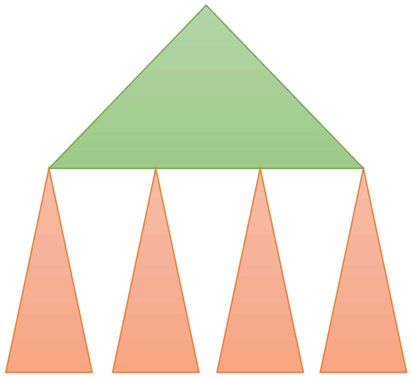
- We use induction on $\mathrm{C}^1(f)$ to prove that $D(f) \le \mathrm{C}^0(f)\mathrm{C}^1(f) \le \mathrm{C}(f)^2$
- If $\mathrm{C}^1(f) = 0$, then $f \equiv 1$ and $\mathrm{D}(f) = 0$

Query Complexity vs. Certificate Complexity

Theorem. For every $f: \{0,1\}^n \to \{0,1\}$, we have $\mathrm{C}(f) \leq \mathrm{D}(f) \leq \mathrm{C}(f)^2$

Proof.

- Suppose $C^1(f) \neq 0$ and we pick any $x \in f^{-1}(0)$. Let S be a smallest certificate for x
- Construct a partial decision tree of depth $|S|$ by querying all the variables in S
- Each leaf ℓ in this partial decision tree corresponds to a function f_ℓ (a restriction of f)
- Since S intersects all the certificates for $y \in f^{-1}(1)$, we have
$$\mathrm{C}^0(f_\ell) \leq \mathrm{C}^0(f), \qquad \mathrm{C}^1(f_\ell) \leq \mathrm{C}^1(f) - 1$$
- By induction, $D(f_\ell) \leq \mathrm{C}^0(f)(\mathrm{C}^1(f) - 1)$ for every ℓ
- Thus, $D(f) \leq \mathrm{C}^0(f) + \mathrm{C}^0(f)(\mathrm{C}^1(f) - 1) = \mathrm{C}^0(f)\mathrm{C}^1(f)$



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Query Complexity vs. Sensitivity

The sensitivity of $f:\{0,1\}^n \to \{0,1\}$ at a point x, denoted by $s_f(x)$, is the number of bits in x such that flipping any one of these bits changes the value of the function:

$$\mathrm{s}_f(x) \coloneqq |\{i \in [n] : f(x) \neq f(x + e_i)\}|$$

- The sensitivity of f is $\mathrm{s}(f) \coloneqq \max_x \mathrm{s}_f(x)$

Example:

- $\mathrm{s}(\mathrm{Parity}) = n$
- $\mathrm{s}(\mathrm{AND}) = n$

Theorem (Nisan-Szegedy '94**).** Every $f:\{0,1\}^n \to \{0,1\}$ satisfies $\deg(f) \geq \sqrt{s(f)/2}$

This result is proved using the "Polynomial Method"

Detour: Degree of Univariate Polynomials

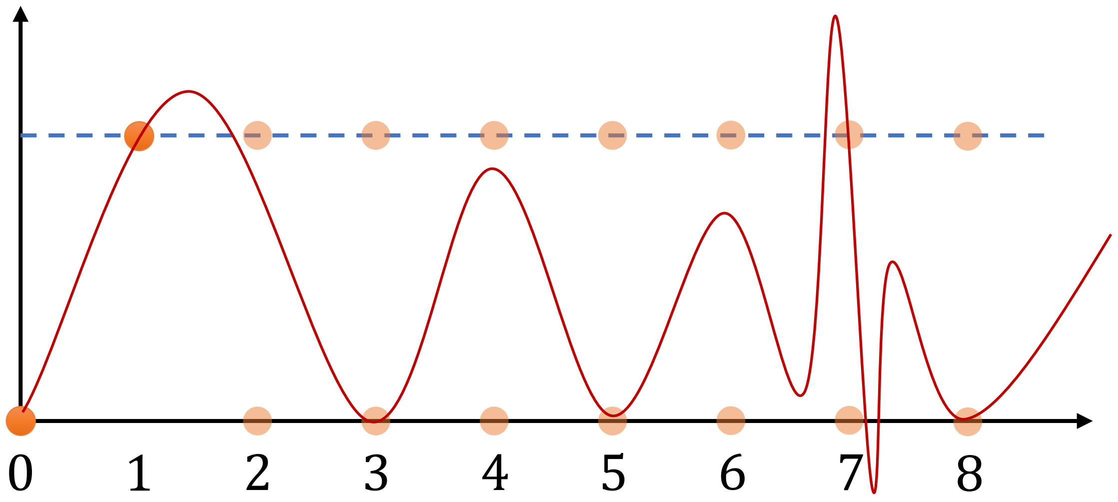
Theorem (Markov Brothers' Inequality). Let $q: \mathbb{R} \to \mathbb{R}$ be a univariate polynomial of degree d such that any real number $x \in [a_1, a_2]$ satisfies $q(x) \in [b_1, b_2]$. Then

$$|q'(x)| \leq d^2 \frac{b_2 - b_1}{a_2 - a_1} \qquad \forall x \in [a_1, a_2]$$

Theorem. Let $q: \mathbb{R} \to \mathbb{R}$ be a univariate polynomial such that

1. $q(0) = 0$ and $q(1) = 1$
2. $0 \leq q(k) \leq 1$ for every $k \in [m]$

Then, $\deg(q) \geq \sqrt{m/2}$



Detour: Degree of Univariate Polynomials

Theorem. Let $q: \mathbb{R} \to \mathbb{R}$ be a univariate polynomial such that: $q(0) = 0$, $q(1) = 1$, and $0 \leq q(k) \leq 1$ for every $k \in [m]$. Then $\deg(q) \geq \sqrt{m/2}$

Proof.

- By the mean value theorem, there exists a point $x \in [0,1]$ with $|q'(x)| \geq 1$
- Let $c \coloneqq \max_{x \in [0,m]} |q'(x)|$, $c \geq 1$
- Since $0 \leq q(k) \leq 1$ for every $k \in [m]$, for every real number $x \in [0, m]$, the mean value theorem implies that $0 - c/2 \leq q(x) \leq 1 + c/2$
- Markov brothers' inequality: $c \leq d^2 \frac{1+c/2-(-c/2)}{m-0} = d^2 \frac{1+c}{m}$
- Thus, we get that $d \geq \sqrt{m \frac{c}{1+c}} \geq \sqrt{\frac{m}{2}}$ since $c \geq 1$

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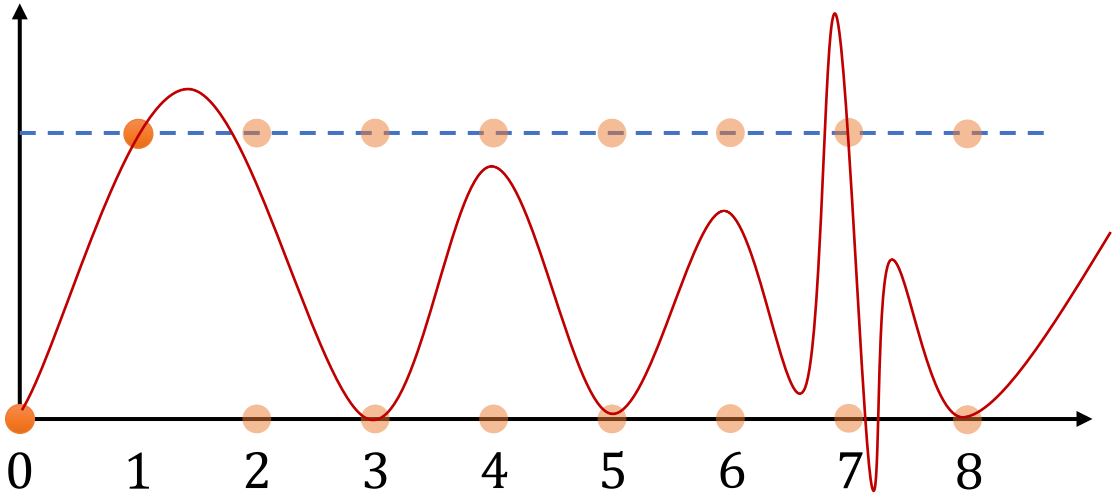
Detour: Degree of Univariate Polynomials



Degree lower bound

$f: \{0,1\}^n \to \{0,1\}$

$f(x) = \sum_S a_S x^S : \mathbb{R}^n \to \mathbb{R}$

$0 \le f(x) \le 1 \ \forall x \in \{0,1\}^n$

?

Detour: Symmetrization

We can symmetrize a multivariate polynomial $p: \mathbb{R}^n \to \mathbb{R}$ by averaging it over all permutations of the variables

$$p^{\mathrm{sym}}(x_1, \dots x_n) \coloneqq \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} p\left(x_{\pi(1)}, \dots, x_{\pi(n)}\right)$$

- $\deg(p^{\mathrm{sym}}) \leq \deg(p)$

Theorem (Minsky-Paper '88**).** For every multilinear polynomial $p: \mathbb{R}^n \to \mathbb{R}$, there is a univariate polynomial $q: \mathbb{R} \to \mathbb{R}$ with $\deg(q) \leq \deg(p)$ such that

$$p^{\mathrm{sym}}(x_1, \dots x_n) = q(x_1 + \dots + x_n) \quad \forall x \in \{0,1\}^n$$

Detour: Symmetrization

Theorem (Minsky-Paper '88). For every multilinear polynomial $p: \mathbb{R}^n \to \mathbb{R}$, there is a univariate polynomial $q: \mathbb{R} \to \mathbb{R}$ with $\deg(q) \le \deg(p)$ such that

$$p^{\mathrm{sym}}(x_1, \ldots x_n) = q(x_1 + \cdots + x_n) \quad \forall x \in \{0,1\}^n$$

Proof.

- Define the k-th elementary symmetric polynomial $e_k(x): \mathbb{R}^n \to \mathbb{R}$ as:

$$e_k(x) := \sum_{S \subseteq [n]:|S|=k} x^S$$

- $\{e_0, \cdots, e_n\}$ forms a basis for symmetric polynomials
- Thus, $p^{\mathrm{sym}} = c_0 + c_1 e_1 + \cdots + c_d e_d$, where $\deg(p^{\mathrm{sym}}) = d$
- Notice that if $x \in \{0,1\}^n$, then $e_k(x) = \binom{x_1 + \cdots + x_n}{k}$, and we have

$$p^{\mathrm{sym}}(x) = \underbrace{c_0 + c_1 \binom{t}{1} + \cdots + c_d \binom{t}{d}}_{q(t)}, \qquad t := x_1 + \cdots + x_n \quad \forall x \in \{0,1\}^n$$

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Detour: Symmetrization

Corollary. Let $f: \{0,1\}^n \to \{0,1\}$ satisfy $f(0, \ldots, 0) = 0$ and $f(e_1) = f(e_2) = \cdots = f(e_n) = 1$. Then $\deg(f) \geq \sqrt{n/2}$

Proof.

- Let $p^{\text{sym}}: \mathbb{R}^n \to \mathbb{R}$ be the symmetrization of f. Then $\deg(p^{sym}) \leq \deg(f)$
- Let $q: \mathbb{R} \to \mathbb{R}$ be the univariate polynomial provided by the previous theorem
- $q(0) = p^{\text{sym}}(0, \ldots, 0) = f(0, \ldots, 0) = 0$
- $q(1) = p^{\text{sym}}(e_1) = \frac{f(e_1)+\cdots+f(e_n)}{n} = 1$
- $q(k) \in [0,1]$ for all $k \in [n]$
- Polynomial method implies that $\deg(q) \geq \sqrt{n/2}$

$\blacksquare$

Query Complexity vs. Sensitivity

The sensitivity of $f: \{0,1\}^n \to \{0,1\}$ at a point x, denoted by $s_f(x)$, is the number of bits in x such that flipping any one of these bits changes the value of the function:

$$s_f(x) \coloneqq |\{i \in [n] : f(x) \neq f(x + e_i)\}|$$

- The sensitivity of f is $\mathrm{s}(f) \coloneqq \max_x \mathrm{s}_f(x)$

Theorem (Nisan-Szegedy ’94). Every $f: \{0,1\}^n \to \{0,1\}$ satisfies $\deg(f) \geq \sqrt{\mathrm{s}(f)/2}$

Proof.

- Let $y \in \{0,1\}^n$ with $s \coloneqq \mathrm{s}_f(y) = \mathrm{s}(f)$, and let $i_1, \ldots, i_s \in [n]$ be the sensitive coordinates
- We may assume $f(y) = 0$
- Define $g: \{0,1\}^s \to \{0,1\}$ as $g(z) \coloneqq f(y + z_1 e_{i_1} + \cdots + z_s e_{i_s})$

Query Complexity vs. Sensitivity

Theorem (Nisan-Szegedy ’94). Every $f: \{0,1\}^n \to \{0,1\}$ satisfies $\deg(f) \geq \sqrt{s(f)/2}$

Proof.

- Let $y \in \{0,1\}^n$ with $s \coloneqq s_f(y) = s(f)$, and let $i_1, \dots, i_s \in [n]$ be the sensitive coordinates
- We may assume $f(y) = 0$
- Define $g: \{0,1\}^s \to \{0,1\}$ as $g(z) \coloneqq f(y + z_1 e_{i_1} + \cdots + z_s e_{i_s})$

$$g(z) = f(x_1, \dots, x_n), \qquad x_i \coloneqq \begin{cases} y_i & \text{if } i \notin \{i_1, \dots, i_s\} \\ z_i & \text{if } i \in \{i_1, \dots, i_s\}, y_i = 0 \\ 1 - z_i & \text{if } i \in \{i_1, \dots, i_s\}, y_i = 1 \end{cases}$$

- $\deg(g) \leq \deg(f)$, $g(0) = 0$, $g(e_i) = 1$ (by the definition of sensitivity)
- Thus, $\deg(g) \geq \sqrt{s/2}$

∎

Query Complexity vs. Block Sensitivity

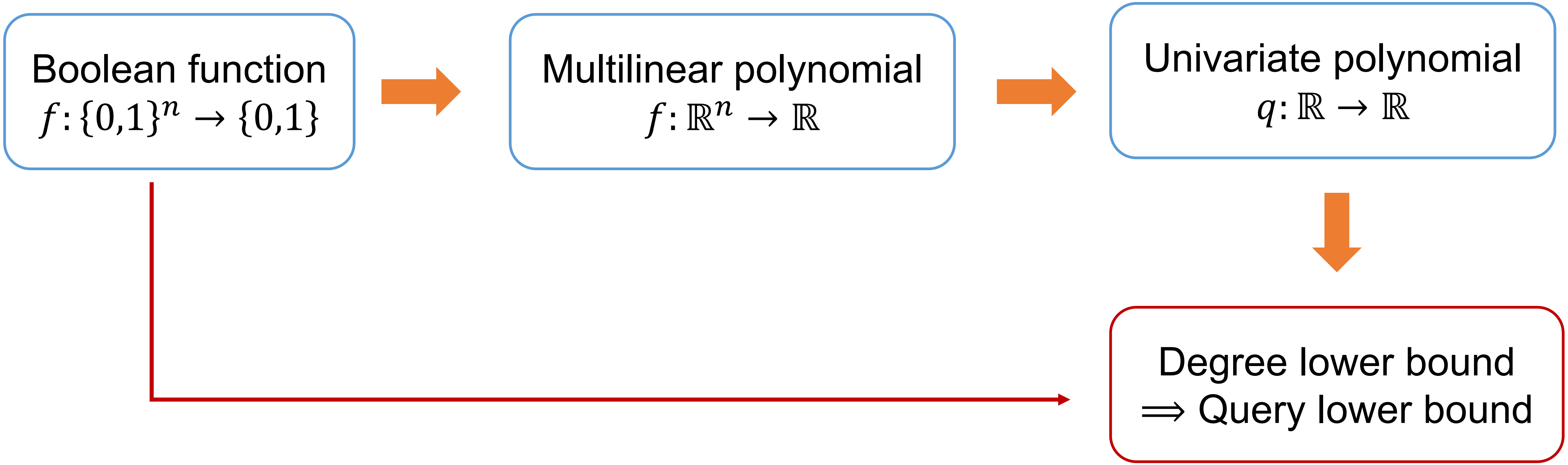
The block sensitivity of $f: \{0,1\}^n \to \{0,1\}$ at a point x, denoted by $\mathrm{bs}_f(x)$, is the maximum number of disjoint subsets $B_1, \dots, B_k \subseteq [n]$ such that $f(x) \neq f\left(x + \mathbf{1}_{B_i}\right)$ for all $i \in [k]$

- The block sensitivity of f is $\mathrm{bs}(f) \coloneqq \max_x \mathrm{bs}_f(x)$
- $\mathrm{bs}(f) \geq s(f)$ (we can take each block of size 1)

Theorem (Nisan-Szegedy '94**).** Every $f: \{0,1\}^n \to \{0,1\}$ satisfies $\deg(f) \geq \sqrt{\mathrm{bs}(f)/2}$

- The proof is almost the same as the previous theorem

Polynomial Method



Query Complexity vs. Block Sensitivity

Theorem (Nisan '91). Every function $f: \{0,1\}^n \to \{0,1\}$ satisfies $\mathrm{C}(f) \leq \mathrm{s}(f)\mathrm{bs}(f)$

Proof.

- Let $x \in \{0,1\}^n$ with a maximal set of sensitive blocks $B_1, \dots, B_s \subseteq [n]$ and $s \leq \mathrm{bs}(f)$
- Wlog, we may assume that B_i is minimal: $\forall j \in B_i$, $f(x) = f\left(x + \mathbf{1}_{B_i \setminus \{j\}}\right)$
- Thus, on the input $x + \mathbf{1}_{B_i}$, each coordinate in B_i is sensitive. Hence, $|B_i| \leq \mathrm{s}(f)$
- Since $\{B_1, \dots, B_s\}$ is maximal, fixing the values of the variables in $B = B_1 \cup \dots \cup B_s$ must determine the value of $f(x)$; otherwise, we could add one more sensitive block
- Thus, $\mathrm{C}_f(x) \leq |B| = |B_1| + \dots + |B_s| \leq s(f)\mathrm{bs}(f)$ for any $x \in \{0,1\}^n$ ∎

Corollary. For any $f: \{0,1\}^n \to \{0,1\}$, we have

$$\deg(f) \leq \mathrm{D}(f) \leq \mathrm{s}(f)^2 \mathrm{bs}(f)^2 \leq \mathrm{bs}(f)^4 \lesssim \deg(f)^8$$

Randomized Query Complexity

A randomized decision tree $\mathbf{T}$ of depth d is a random variable that takes values in the set of decision trees of depth at most d

The randomized query complexity of $f: \{0,1\}^n \to \{0,1\}$, denoted by $\mathrm{R}(f)$, is the smallest d such that there exists a randomized decision tree T of depth d such that

$$\Pr_{\mathbf{T}}[\mathbf{T}(x) \neq f(x)] \leq \frac{1}{3} \qquad \forall x \in \{0,1\}^n$$

- Define $g(x) \coloneqq \mathbb{E}[\mathbf{T}(x)] = \sum_T \mu(T) T(x)$, where μ is the probability distribution of $\mathbf{T}$
- $\|f - g\|_\infty \leq 1/3$ since

$$\|f - g\|_\infty = \max_x \left| f(x) - \sum_T \mu(T) T(x) \right| \leq \max_x \sum_T \mu(T) |f(x) - T(x)|$$
$$= \max_x \Pr[\mathbf{T}(x) \neq f(x)] \leq 1/3$$

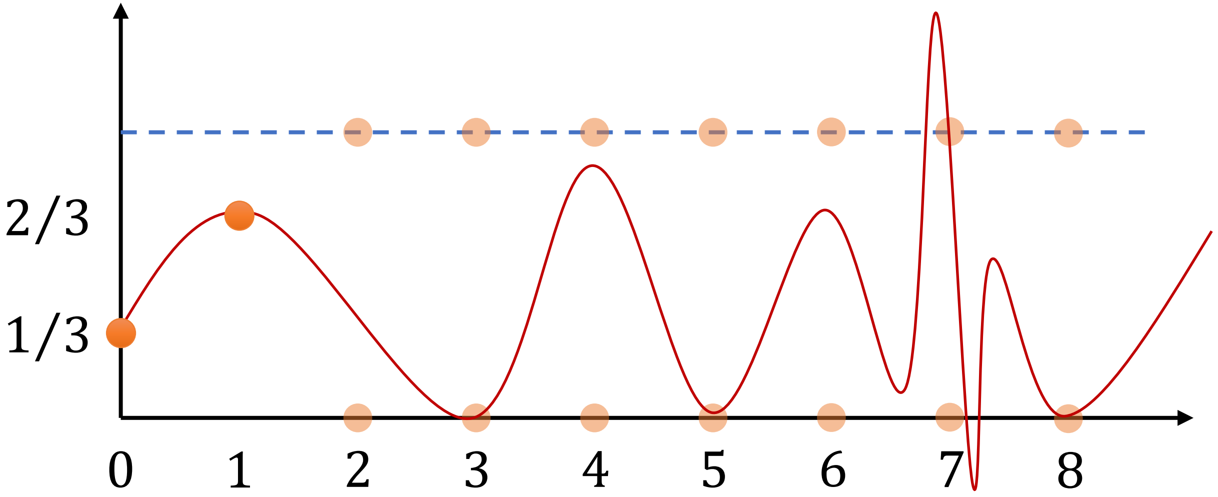
- Since $\deg(f) \leq D(f)$, we have $\deg(T) \leq d$ and $\deg(g) \leq d$

Randomized Query Complexity vs. Approximate Degree

The approximate degree of f, denoted by $\widetilde{\deg}(f)$, is the smallest d such that there exists $g: \{0,1\}^n \to \mathbb{R}$ with $\deg(g) \leq d$ and $\|f - g\|_\infty \leq 1/3$

Proposition. For every $f: \{0,1\}^n \to \{0,1\}$, we have $\widetilde{\deg}(f) \leq \mathrm{R}(f)$

Theorem (Nisan-Szegedy '94). Every function $f: \{0,1\}^n \to \{0,1\}$ satisfies $\widetilde{\deg}(f) \geq \sqrt{\mathrm{bs}(f)/6}$



$\deg(q) \geq \sqrt{m/6}$

Randomized Query Complexity vs. Approximate Degree

The approximate degree of f, denoted by $\widetilde{\deg}(f)$, is the smallest d such that there exists $g: \{0,1\}^n \to \mathbb{R}$ with $\deg(g) \leq d$ and $\|f - g\|_\infty \leq 1/3$

Proposition. For every $f: \{0,1\}^n \to \{0,1\}$, we have $\widetilde{\deg}(f) \leq \mathrm{R}(f)$

Theorem (Nisan-Szegedy '94). Every function $f: \{0,1\}^n \to \{0,1\}$ satisfies $\widetilde{\deg}(f) \geq \sqrt{\mathrm{bs}(f)/6}$

- Since $\mathrm{D}(f) \geq \mathrm{R}(f)$ and $\mathrm{D}(f) \leq \mathrm{bs}(f)^4$, we have

$$\mathrm{D}(f)^{\frac{1}{8}} \lesssim \sqrt{\mathrm{bs}(f)} \lesssim \widetilde{\deg}(f) \leq \mathrm{R}(f) \leq \mathrm{D}(f)$$

The following measures of Boolean functions are polynomially equivalent:

- Degree $\deg(f)$
- Certificate complexity $\mathrm{C}(f)$
- Block sensitivity $\mathrm{bs}(f)$
- Approximate degree $\widetilde{\deg}(f)$
- Deterministic query complexity $\mathrm{D}(f)$
- Randomized query complexity $\mathrm{R}(f)$

Can we add the sensitivity $\mathrm{s}(f)$ to this list?

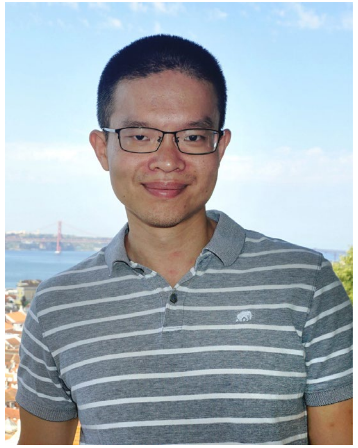
The Sensitivity Conjecture

- (Nisan-Szegedy ’94) conjectured that the sensitivity of a Boolean function is polynomially equivalent to the degree, block sensitivity, query complexity, …
- A quote from Scott Aaronson:

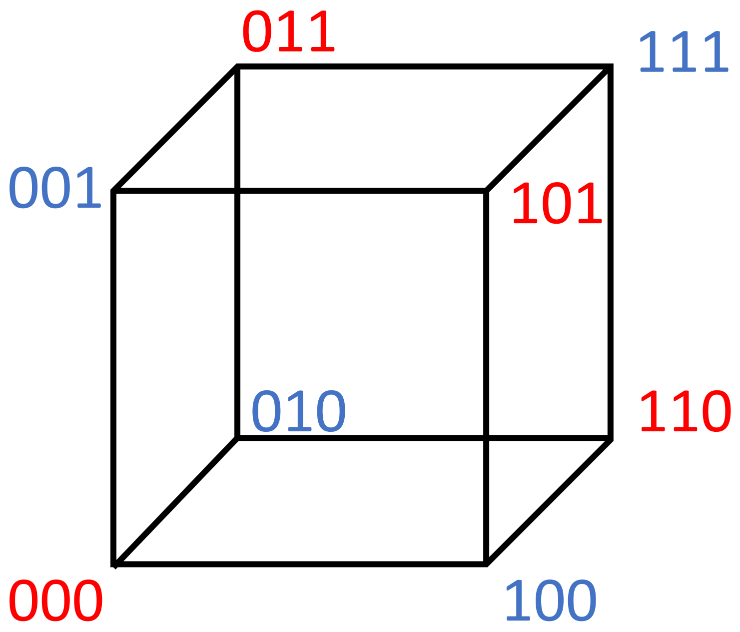
  Ever since it was posed by Nisan and Szegedy in 1989, this conjecture has stood as one of the most frustrating and embarrassing open problems in all of combinatorics and theoretical computer science. It seemed so easy, and so similar to other statements that had 5-line proofs. But a lot of the best people in the field sank months into trying to prove it.

- Resolved by Hao Huang in 2019 (a beautiful 6-page paper)

Theorem. Every function $f: \{0,1\}^n \to \{0,1\}$ satisfies $s(f) \geq \sqrt{\deg(f)}$



The Sensitivity Theorem

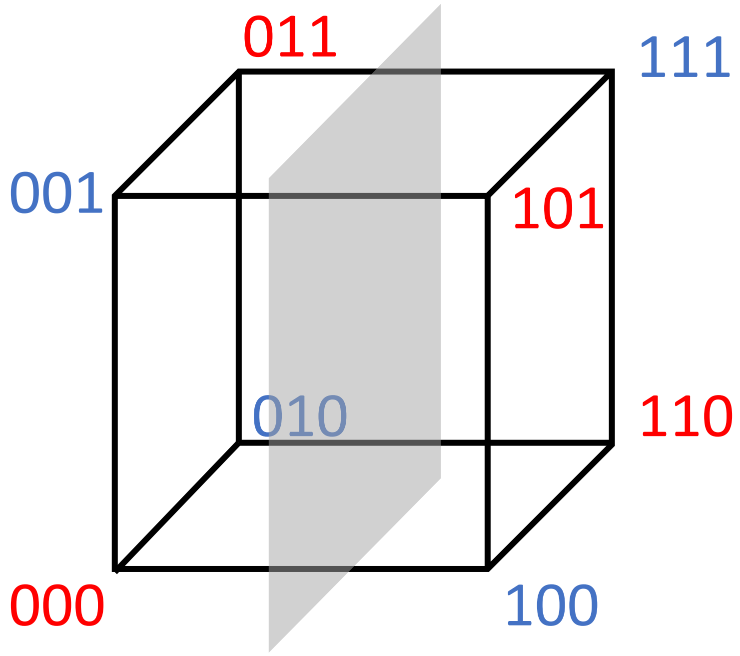


Define a bipartition of $\{0,1\}^n$:

$$V_{\text{even}} := \left\{x \in \{0,1\}^n : \sum_{i\in[n]} x_i \equiv 0 \bmod 2\right\}, \qquad V_{\text{odd}} := \left\{x \in \{0,1\}^n : \sum_{i\in[n]} x_i \equiv 1 \bmod 2\right\}$$

Notice that the Boolean hypercube Q_n is a bipartite graph between V_{even} and V_{odd}

The Sensitivity Theorem



Adjacency matrix $\boldsymbol{A}_n$ of Q_n:

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \boldsymbol{A}_n = \begin{bmatrix} \boldsymbol{A}_{n-1} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{A}_{n-1} \end{bmatrix}$$

The Sensitivity Theorem

Theorem (Huang '19**).** For every $T \subseteq \{0,1\}^n$ with $|T| > 2^{n-1}$, let $\mathcal{H} \coloneqq Q_n[T]$ be the subgraph induced by Q_n on T. Then, the maximum degree of $\mathcal{H}$ satisfies $\Delta(\mathcal{H}) \geq \sqrt{n}$

Proof.

- Define the signed adjacency matrix $\widetilde{\boldsymbol{A}}_n$:

$$\widetilde{\boldsymbol{A}}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \widetilde{\boldsymbol{A}}_n = \begin{bmatrix} \widetilde{\boldsymbol{A}}_{n-1} & \boldsymbol{I} \\ \boldsymbol{I} & -\widetilde{\boldsymbol{A}}_{n-1} \end{bmatrix}$$

- Let $\boldsymbol{B}$ denote the $T \times T$ principal submatrix of $\widetilde{\boldsymbol{A}}_n$
- We have

$$\lambda_1(\boldsymbol{B}) \leq \max_{i \in T} \sum_{j \in T} |\boldsymbol{B}_{i,j}| = \max_{i \in T} \sum_{j \in T} |\widetilde{\boldsymbol{A}}_{i,j}| = \max_{i \in T} \sum_{j \in T} |\boldsymbol{A}_{i,j}| = \max_{i \in T} \deg_{\mathcal{H}}(i) \leq \Delta(\mathcal{H})$$

(why?)

The Sensitivity Theorem

Theorem (Huang ’19**).** For every $T \subseteq \{0,1\}^n$ with $|T| > 2^{n-1}$, let $\mathcal{H} \coloneqq \mathcal{Q}_n[T]$ be the subgraph induced by $\mathcal{Q}_n$ on T. Then, the maximum degree of $\mathcal{H}$ satisfies $\Delta(\mathcal{H}) \geq \sqrt{n}$

Proof.

- Define the signed adjacency matrix $\widetilde{\boldsymbol{A}}_n$:

$$\widetilde{\boldsymbol{A}}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \widetilde{\boldsymbol{A}}_n = \begin{bmatrix} \widetilde{\boldsymbol{A}}_{n-1} & \boldsymbol{I} \\ \boldsymbol{I} & -\widetilde{\boldsymbol{A}}_{n-1} \end{bmatrix}$$

- We can prove by induction that $\left(\widetilde{\boldsymbol{A}}_n\right)^2 = n\boldsymbol{I}$:

$$\begin{bmatrix} \widetilde{\boldsymbol{A}}_{n-1} & \boldsymbol{I} \\ \boldsymbol{I} & -\widetilde{\boldsymbol{A}}_{n-1} \end{bmatrix} \cdot \begin{bmatrix} \widetilde{\boldsymbol{A}}_{n-1} & \boldsymbol{I} \\ \boldsymbol{I} & -\widetilde{\boldsymbol{A}}_{n-1} \end{bmatrix} = \begin{bmatrix} \left(\widetilde{\boldsymbol{A}}_{n-1}\right)^2 + \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \left(\widetilde{\boldsymbol{A}}_{n-1}\right)^2 + \boldsymbol{I} \end{bmatrix}$$

- Thus, $\lambda_i\left(\widetilde{\boldsymbol{A}}_n\right) = \sqrt{n}$ or $-\sqrt{n}$ for any $i \in [N]$
- Since $\mathrm{tr}\left[\widetilde{\boldsymbol{A}}_n\right] = 0$, we know that 2^{n-1} eigenvalues are $\sqrt{n}$ and 2^{n-1} eigenvalues are $-\sqrt{n}$

Matrix Analysis: Eigenvalue Inequalities (Lecture 15)

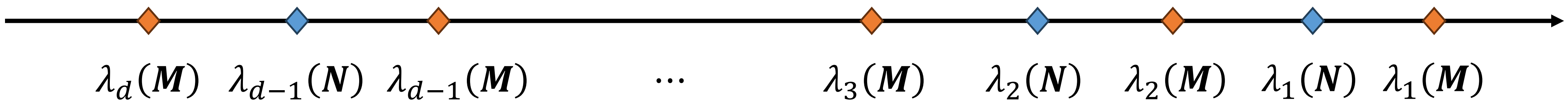
Theorem (Cauchy Interlacing Theorem). For any $\boldsymbol{M} \in \mathbb{S}^{d\times d}$ and $\boldsymbol{V} \in \mathbb{R}^{d\times r}$ with $\boldsymbol{V}^\top \boldsymbol{V} = \boldsymbol{I}_r$,

$$\lambda_{d-r+k}(\boldsymbol{M}) \le \lambda_k(\boldsymbol{V}^\top \boldsymbol{M}\boldsymbol{V}) \le \lambda_k(\boldsymbol{M}) \quad \forall k \in [r]$$

Corollary. For any $\boldsymbol{M} \in \mathbb{S}^{d\times d}$ and $\boldsymbol{N} \in \mathbb{S}^{r\times r}$ a principal submatrix of $\boldsymbol{M}$,

$$\lambda_{d-r+k}(\boldsymbol{M}) \le \lambda_k(\boldsymbol{N}) \le \lambda_k(\boldsymbol{M}) \quad \forall k \in [r]$$

- If $r = d - 1$, then the eigenvalues of $\boldsymbol{N}$ interlace the eigenvalues of $\boldsymbol{M}$



The Sensitivity Theorem

Theorem (Huang '19**).** For every $T \subseteq \{0,1\}^n$ with $|T| > 2^{n-1}$, let $\mathcal{H} \coloneqq Q_n[T]$ be the subgraph induced by Q_n on T. Then, the maximum degree of $\mathcal{H}$ satisfies $\Delta(\mathcal{H}) \geq \sqrt{n}$

Proof.

- Let $\boldsymbol{B}$ denote the $T \times T$ principal submatrix of $\widetilde{\boldsymbol{A}}_n$

$$\Delta(\mathcal{H}) \geq \lambda_1(\boldsymbol{B})$$

- Thus, $\lambda_i(\widetilde{\boldsymbol{A}}_n) = \sqrt{n}$ or $-\sqrt{n}$ for any $i \in [N]$
- Since $\mathrm{tr}[\widetilde{\boldsymbol{A}}_n] = 0$, we know that half of the eigenvalues are $\sqrt{n}$ and half are $-\sqrt{n}$
- Cauchy interlacing theorem gives that

$$\lambda_1(\boldsymbol{B}) \geq \lambda_{2^n-|T|+1}(\widetilde{\boldsymbol{A}}_n) \geq \lambda_{2^{n-1}}(\widetilde{\boldsymbol{A}}_n) = \sqrt{n}$$

■

The Sensitivity Theorem

Theorem (The Sensitivity Theorem). Every function $f: \{0,1\}^n \to \{0,1\}$ satisfies $\mathrm{s}(f) \geq \sqrt{\deg(f)}$

Proof (via the reduction given by (Gotsman-Linial '92)).

- We may assume $\deg(f) = n$; otherwise, we can pick a $\deg(f)$-degree monomial in f and restrict the remaining variables to 0. Let g be the restricted function. Then, $\deg(g) = \deg(f)$ (i.e., full degree) and $\mathrm{s}(g) \leq \mathrm{s}(f)$. Thus, if we prove that $\mathrm{s}(g) \geq \sqrt{\deg(g)}$, the we also have $\mathrm{s}(f) \geq \sqrt{\deg(f)}$

The Sensitivity Theorem

Proof (via a reduction given by (Gotsman-Linial '92)).

- We change the output domain of f from $\{0,1\}$ to $\{-1,1\}$, and we can write the its Fourier transform
$$f(x) = \sum_S \hat{f}(S)\chi_S(x)$$
- Since $\deg(f) = n$, $\hat{f}(\mathbf{1}) \neq 0$. We define $h(x) \coloneqq f(x)\chi_{\mathbf{1}}(x) = f(x)(-1)^{x_1+\cdots+x_n}$
- We have $\hat{h}(S) = \langle f\chi_{\mathbf{1}}, \chi_S\rangle = \hat{f}(\mathbf{1}+S)$. Thus, $\mathbb{E}[h] = \hat{h}(\mathbf{0}) = \hat{f}(\mathbf{1}) \neq 0$
- For any $i \in [n]$, $f(x) \neq f(x+e_i) \Leftrightarrow h(x) = h(x+e_i)$. Thus, $\mathrm{s}_f(x) = |\{i \in [n] : h(x) = h(x+e_i)\}|$
- That is, $\mathrm{s}_f(x) = \deg_{Q_n[T]}(x)$ where $T \coloneqq \{y \in \{0,1\}^n : h(y) = h(x)\}$
- Since $|h^{-1}(1)| \neq |h^{-1}(-1)|$, we have $T > 2^{n-1}$ for some $x \in \{0,1\}^n$
- By Huang's theorem, $\Delta(Q_n[T]) \geq \sqrt{n}$. Therefore, $\mathrm{s}(f) \geq \sqrt{n} = \sqrt{\deg(f)}$

■

Table 1: Best known separations between complexity measures

	D	R_0	R	C	RC	bs	s	λ	Q_E	deg	Q	$\widetilde{\deg}$
D		2, 2 [ABB+17]	2, 3 [ABB+17]	2, 2 $\wedge \circ \vee$	2, 3 $\wedge \circ \vee$	2, 3 $\wedge \circ \vee$	3, 6 [BHT17]	4, 6 [ABB+17]	2, 3 [ABB+17]	2, 3 [GPW18]	4, 4 [ABB+17]	4, 4 [ABB+17]
R_0	1, 1 $\oplus$		2, 2 [ABB+17]	2, 2 $\wedge \circ \vee$	2, 3 $\wedge \circ \vee$	2, 3 $\wedge \circ \vee$	3, 6 [BHT17]	4, 6 [ABB+17]	2, 3 [ABB+17]	2, 3 [GJPW18]	3, 4 [ABB+17]	4, 4 [ABB+17]
R	1, 1 $\oplus$	1, 1 $\oplus$		2, 2 $\wedge \circ \vee$	2, 3 $\wedge \circ \vee$	2, 3 $\wedge \circ \vee$	3, 6 [BHT17]	4, 6 [ABB+17]	$\frac{3}{2}$, 3 [ABB+17]	2, 3 [GJPW18]	3, 4 [BS20] [SSW20]	4, 4 [ABB+17]
C	1, 1 $\oplus$	1, 1 $\oplus$	1, 2 $\oplus$		2, 2 [GSS13]	2, 2 [GSS13]	2.22, 5 [BHT17]	2.44, 6 [BHT17][a]	1.15, 3 [Amb13]	1.63, 3 [NW95]	2, 4 $\wedge$	2, 4 $\wedge$
RC	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$		$\frac{3}{2}$, 2 [GSS13]	2, 4 [Rub95]	2, 4 $\wedge$	1.15, 2 [Amb13]	1.63, 2 [NW95]	2, 2 $\wedge$	2, 2 $\wedge$
bs	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$		2, 4 [Rub95]	2, 4 $\wedge$	1.15, 2 [Amb13]	1.63, 2 [NW95]	2, 2 $\wedge$	2, 2 $\wedge$
s	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$		2, 2 $\wedge$	1.15, 2 [Amb13]	1.63, 2 [NW95]	2, 2 $\wedge$	2, 2 $\wedge$
λ	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$		1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$
Q_E	1, 1 $\oplus$	1.33, 2 $\bar{\wedge}$-tree	1.33, 3 $\bar{\wedge}$-tree	2, 2 $\wedge \circ \vee$	2, 3 $\wedge \circ \vee$	2, 3 $\wedge \circ \vee$	3, 6 [BHT17]	4, 6 [ABK16]		2, 3 [ABK16]	2, 4 $\wedge$	4, 4 [ABK16]
deg	1, 1 $\oplus$	1.33, 2 $\bar{\wedge}$-tree	1.33, 2 $\bar{\wedge}$-tree	2, 2 $\wedge \circ \vee$	2, 2 $\wedge \circ \vee$	2, 2 $\wedge \circ \vee$	2, 2 $\wedge \circ \vee$	2, 2 $\wedge$	1, 1 $\oplus$		2, 2 $\wedge$	2, 2 $\wedge$
Q	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	2, 2 [ABK16]	2, 3 [ABK16]	2, 3 [ABK16]	3, 6 [BHT17]	4, 6 [ABK16]	1, 1 $\oplus$	2, 3 [ABK16]		4, 4 [ABK16]
$\widetilde{\deg}$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	2, 2 [BT17]	2, 2 [BT17]	2, 2 [BT17]	2, 2 [BT17]	2, 2 [BT17]	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	

- An entry a, b in the row M_1 and column M_2 roughly means that there exists a function g with $M_1(g) \geq M_2(g)^{a-o(1)}$, and for all total functions f, $M_1(f) \leq M_2(f)^{b+o(1)}$ (see [ABK16]

(Aaronson-Ben-David-Kothari-Rao-Tal '21)